Original Scientific Article Journal of Decision Analytics and Intelligent Computing
Vol. 4 issue 1, (2024) 241-252 https://doi.org/10.31181/jdaic10021122024a

Solution of second order linear homogeneous fuzzy difference equation with constant coefficients by geometric approach

Abdul Alamin[1], Kamal Hossain Gazi[1] and Sankar Prasad Mondal[1,*]

[1] Department of Applied Mathematics, Maulana Abul Kalam Azad University of Technology, West Bengal, India

* Correspondence: sankar.res07@gmail.com

Received 24 July 2024
Accepted for publication 10 December 2024
Published 21 December 2024

Abstract

In this article, we have solved an initial valued second-order linear homogeneous fuzzy difference equation with constant coefficients using a geometric approach. General fuzzy solution structures for the three cases are established depending on the auxiliary roots of the corresponding homogeneous difference equation. Finally, we have taken the numerical examples and solved them using the theoretical results, and depicted the graphical scenarios to realize the deviation of the uncertain solution from the exact solution as well as the vagueness of the initial values.

Keywords: second order linear homogeneous difference equation, fuzzy difference equation, geometric approach.

1. Introduction

A difference equation is actually an equation which specifies the change of the variable between two periods. The theory of difference equation occupies an important position in different fields. The theory of difference equations developed greatly during the last three decades. The difference equations are solved in fuzzy (Pamucar et al., 2022; Hussain and Ullah, 2024) and neutrosophic (Pamucar et al., 2020) environments to the capability of uncertainty dealing. Zadeh (1965) first introduced the fuzzy set theory and its fundamental properties.

1.1 A brief literature study

Nowadays, the study of the qualitative behaviour of difference equations or a system of difference equations is a topic of great interest in uncertain scenarios (Abid and Saqlain, 2024; Kizielewicz and Sałabun, 2024). The researchers Deeba and De Korvin (1999), Lakshmikantham and Vatsala (2002) and Alamin et al. (2020) have discussed the few first order fuzzy difference equations in the context of theoretical aspects and their application in reality. Although first order linear difference equations have been used to demonstrate the various discrete dynamical models in the fuzzy environment (see: Khastan, 2018; Mahmoodirad and Niroomand, 2023; Özdağoğlu et al., 2024), work on second order fuzzy difference equations is very rare. Various properties of fuzzy numbers are described in various studies, including Pamucar and Ćirović (2018) and Riaz et al. (2021). Alamin et al. (2025)

have solved second order linear difference equations by intuitionistic fuzzy extension principle and Lagrange multiplier method in a fuzzy environment, respectively. Among the various techniques or methodologies, the geometric approach is one of the methodologies to solve a fuzzy difference equation. Gasilov et al. (2011; 2014; 2015) studied this method through some articles regarding linear differential equations (both the boundary value problem and the initial value problem). Recently, Alamin et al. (2025) have solved a first order non-homogeneous difference equation. We try to solve a second order homogeneous difference equation with a constant coefficient using this method.

1.2 Motivation and Novelties

On the basis of a short literature study of fuzzy difference equations, the second order linear fuzzy difference equations are solved using a very limited technique or methodology. Thus, we are interested in solving the second order fuzzy linear difference equations using a geometric approach, which was not done earlier. We have given a general structure of the solution depending on the auxiliary roots of the homogeneous crisp problem. Numerical illustration and graphical configuration have been done.

1.3 Structure of the study

This section discussed the structure of this study in detail. The introduction section is described in Section 1. In the rest of the section, Section 2, the necessary preliminaries are given. In Section 3, the general structure of the solution and in Section 4, numerical illustration is drawn. Finally, the conclusion is given in Section 5.

2. Preliminaries

This section discusses the preliminaries of the mathematical tool, i.e., fuzzy sets and their properties. A fuzzy set is written in an ordered pair where 1st entry is an element itself and the latter is their degree of membership value. The definition and extensions of the fuzzy set are described as follows:

2.1 Fuzzy Set

The fuzzy set was first introduced by Zadeh (1965; 1972; 1973). Unlike classical set theory, each entry either belong to the set or doesn't belong to the set, i.e., the degree of membership value is binary, $\{0,1\}$ (Chakraborty et al., 2024; Pamučar et al., 2011a). In the classical set theory, this binary characterization is not often too rigid a model for real-world scenarios where the boundaries are unclear (Tešić et al., 2024; Sarfraz, 2024). For example, describing 'good character', 'excellent result', or 'poor health' is subjective and varies by instance.

Fuzzy sets have the flexibility to define unclear boundaries by their degree of membership value (Ayub et al., 2022). In a fuzzy set, every element is assigned with real numbers called the degree of membership value, which always belongs to $[0,1]$ (Gazi et al., 2024). The value of the membership degree is called the membership function of the fuzzy set to control the belongingness of the element in the set (Wang et al. 2024; Kamran et al., 2024). The membership values 0 describe the element not in the set, 1describe the element that fully belongs to the set and any intermediate value in $(0,1)$ describe the elements that partially belong to the set (Božanic et al., 2023). Fuzzy sets are applied in various fields, including differential equations (Gazi et al., 2024), difference equations (Alamin et al., 2020), series solutions (Singh et al., 2024b), integral transformation (Singh et al., 2024a), etc. The definition and properties are defined as follows:

Definition 1: [Fuzzy set] (Singh et al., 2024a) A fuzzy set $\tilde{A}$ is defined as a set of ordered pair $\tilde{A} = (s, \mu_{\tilde{A}}(s))$, where S is a nonempty universal set and $s \in S$, A is the classical set. The function $\mu_{\tilde{A}}(s): S \to [0,1]$ is called the membership function and $\mu_{\tilde{A}}(s)$ is the grade of membership of $s \in S$ in $\tilde{A}$.

The membership function $\left(\mu_{\tilde{A}}(s)\right)$ always lies in $[0,1]$ and every element of the fuzzy set is assigned a membership value and written in ordered pairs. The parametric representation of the fuzzy set is α −cut set, where every fuzzy set is denoted by a classical set with respect to α $(0 \leq \alpha \leq 1)$.

Definition 2: [$\boldsymbol{\alpha}$ −cut Set] (Mukherjee et al., 2023) The α −cut of the fuzzy set $\tilde{A}$ of S is given by $A_\alpha = \{s : \mu_{\tilde{A}}(s) \geq \alpha, s \in S, \alpha \in [0,1]\}$.

Definition 3: [Strong $\boldsymbol{\alpha}$ −cut Set] (Mukherjee et al., 2023) The strong α −cut of the fuzzy set $\tilde{A}$ of S is given by $A_\alpha = \{s : \mu_{\tilde{A}}(s) > \alpha, s \in S, \alpha \in [0,1]\}$.

By definition, the α −cut is a crisp set. This is also called the interval of confidence, α-level set, etc.

Definition 4: [Fuzzy number] (Singh et al., 2024b) Consider $\mathbb{R}$ be the set of real numbers is a universal set of discourse. Then, the fuzzy set $\tilde{F}$ define on $\mathbb{R}$ is called the fuzzy number if it satisfies the following conditions:

a) The fuzzy set $(\tilde{F})$ is a normal fuzzy set, i.e., there exist $r \in \mathbb{R}$ such that $\mu_{\tilde{F}}(r) = 1$.

b) The fuzzy set $(\tilde{F})$ is a convex fuzzy set, i.e., $\mu_{\tilde{F}}(\lambda r + (1-\lambda)s) \geq \min_{\forall r,s \in \mathbb{R}} \{\mu_{\tilde{F}}(r), \mu_{\tilde{F}}(s)\}$ where $\lambda \in [0,1]$.

c) The support of the fuzzy set $(\tilde{F})$ must be bounded, i.e., $Support\ (\tilde{F}) = \{r : \mu_{\tilde{F}}(r) > 0\}$ is bounded.

d) The membership function $(\mu_{\tilde{F}})$ of the fuzzy set $(\tilde{F})$ is piecewise continuous.

From the above definitions, we can conclude that all fuzzy numbers are fuzzy sets, but the converse is not always true. Figure 1 represents the fuzzy number (Trapezoidal fuzzy number (TrFN)) and its α −cut graphically.

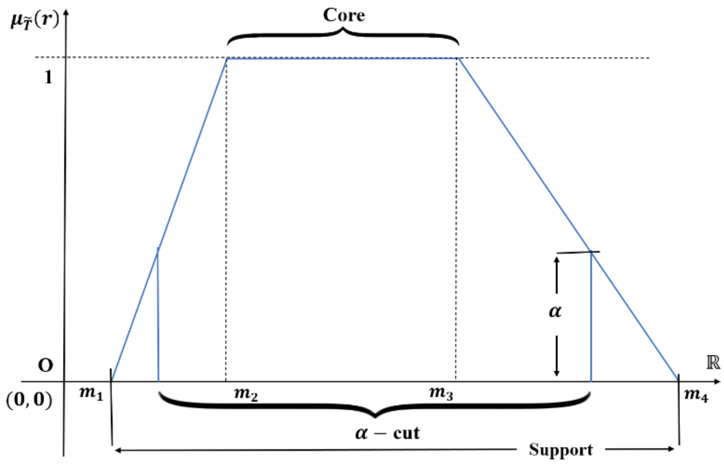

Figure 1. Geometric structure of the fuzzy number

2.2 Triangular Fuzzy Number

This section describes the triangular fuzzy number (TFN) in detail. In TFN, the membership function is triangular in form and the fuzzy number satisfies all of its properties. The TFN is defined as follows:

Definition 5: [Triangular Fuzzy Number (TFN)] (Pamučar et al., 2011b) Assume the set of universal discourse is set of real numbers $(\mathbb{R})$. An ordered triplet number, $\widetilde{T} = (m_1\ m_2, m_3)$ is a triangular fuzzy number on $\mathbb{R}$ and the membership function $(\mu_{\widetilde{T}})$ define as

$$\mu_{\widetilde{T}}(s) = \begin{cases} 0 & for\ s < m_1 \\ \frac{s-m_1}{m_2-m_1} & for\ m_1 \leq s < m_2 \\ 1 & for\ s = m_2 \\ \frac{m_3-s}{m_3-m_2} & for\ m_2 < s \leq m_3 \\ 0 & for\ m_3 < s \end{cases} \quad (1)$$

where $m_1, m_2, m_3 \in \mathbb{R}$ with $m_1 \leq m_2 \leq m_3$ and $s \in \mathbb{R}$.

In the geometric approach, the triangular fuzzy number $\widetilde{T} = (m_1\ m_2, m_3)$ can be expressed as $\widetilde{T} = T_{cp} + \widetilde{T}_{up}$ (certainty part + uncertainty part). The certainty part T_{cp} is the crisp value m_2 where the membership value is always 1. The uncertainty fuzzy number $\widetilde{T}_{up} = (m_1 - m_2, 0,\ m_3 - m_2)$ is a triangular fuzzy number.

2.3 Arithmetic operations on TFNs

The arithmetic operations on TFNs are defined in this section. All the arithmetic operations of TFN are formulated in α −cut forms, which is interval arithmetic (Mukherjee et al., 2023). The arithmetic operations on TFNs are defined as follows:

Consider, $\widetilde{U} = (m_1\ m_2, m_3)$ and $\widetilde{V} = (n_1\ n_2, n_3)$ are two TFNs defined on a universal set $\mathbb{R}$ (set of real numbers). Then the α −cuts of $\widetilde{U}$ and $\widetilde{V}$ are $\widetilde{U}_\alpha = [(m_2 - m_1)\alpha + m_1, -(m_3 - m_2)\alpha + m_3]$ and $\widetilde{V}_\alpha = [(n_2 - n_1)\alpha + n_1, -(n_3 - n_2)\alpha + n_3]$, respectively, where $\alpha \in [0,1]$. Then, arithmetic operations on $\widetilde{U}$ and $\widetilde{V}$ are defined as follows:

(i) Addition of two TFNs:

$$\begin{aligned}\widetilde{U} \oplus \widetilde{V} &= (m_1\ m_2, m_3) \oplus (n_1\ n_2, n_3)\\ &= [(m_2 - m_1)\alpha + m_1, -(m_3 - m_2)\alpha + m_3] \oplus [(n_2 - n_1)\alpha + n_1, -(n_3 - n_2)\alpha + n_3]\\ &= [(m_2 - m_1)\alpha + m_1 + (n_2 - n_1)\alpha + n_1, -(m_3 - m_2)\alpha + m_3 - (n_3 - n_2)\alpha + n_3]\\ &= [\{(m_2 + n_2) - (m_1 + n_1)\}\alpha + (m_1 + n_1), -\{(m_3 + n_3) - (m_2 + n_2)\}\alpha + (m_3 + n_3)]\\ &= \big((m_1 + n_1), (m_2 + n_2), (m_3 + n_3)\big)\end{aligned} \quad (2)$$

(ii) Substruction of two TFNs:

$$\begin{aligned}\widetilde{U} \ominus \widetilde{V} &= (m_1\ m_2, m_3) \ominus (n_1\ n_2, n_3)\\ &= [(m_2 - m_1)\alpha + m_1, -(m_3 - m_2)\alpha + m_3] \ominus [(n_2 - n_1)\alpha + n_1, -(n_3 - n_2)\alpha + n_3]\\ &= \big[(m_2 - m_1)\alpha + m_1 - (-(n_3 - n_2)\alpha + n_3), -(m_3 - m_2)\alpha + m_3 - \big((n_2 - n_1)\alpha + n_1\big)\big]\\ &= [(m_2 - m_1)\alpha + m_1 + (n_3 - n_2)\alpha - n_3, -(m_3 - m_2)\alpha + m_3 - (n_2 - n_1)\alpha - n_1]\\ &= [\{(m_2 - n_2) - (m_1 - n_3)\}\alpha + (m_1 - n_3), -\{(m_3 - n_1) - (m_2 - n_2)\}\alpha + (m_3 - n_1)]\\ &= \big((m_1 - n_3), (m_2 - n_2), (m_3 - n_1)\big)\end{aligned} \quad (3)$$

(iii) Scholar multiplication of TFN:

$$\begin{aligned}\lambda\widetilde{U} &= \lambda \times \widetilde{U} = \lambda \times (m_1\ m_2, m_3)\\ &= \lambda \times [(m_2 - m_1)\alpha + m_1, -(m_3 - m_2)\alpha + m_3]\\ &= [\lambda \times (m_2 - m_1)\alpha + \lambda \times m_1, -\lambda \times (m_3 - m_2)\alpha + \lambda \times m_3]\\ &= [(\lambda \times m_2 - \lambda \times m_1)\alpha + \lambda \times m_1, -(\lambda \times m_3 - \lambda \times m_2)\alpha + \lambda \times m_3]\\ &= (\lambda \times m_1, \lambda \times m_2, \lambda \times m_3)\\ &= (\lambda m_1, \lambda m_2, \lambda m_3)\end{aligned} \quad (4)$$

where λ be a positive scholar number ($\lambda \geq 0$).

(iv) Multiplication of two TFNs:

$$\begin{aligned}\widetilde{U} \otimes \widetilde{V} &= (m_1\ m_2, m_3) \otimes (n_1\ n_2, n_3)\\ &= [(m_2 - m_1)\alpha + m_1, -(m_3 - m_2)\alpha + m_3] \otimes [(n_2 - n_1)\alpha + n_1, -(n_3 - n_2)\alpha + n_3]\\ &= [\min\{P_1, P_2, P_3, P_4\}, \max\{P_1, P_2, P_3, P_4\}]\end{aligned} \quad (5)$$

where $P_1 = \{(m_2 - m_1)\alpha + m_1\} \times \{(n_2 - n_1)\alpha + n_1\}$, $P_2 = \{(m_2 - m_1)\alpha + m_1\} \times \{-(n_3 - n_2)\alpha + n_3\}$, $P_3 = \{-(m_3 - m_2)\alpha + m_3\} \times \{(n_2 - n_1)\alpha + n_1\}$ and $P_4 = \{-(m_3 - m_2)\alpha + m_3\} \times \{-(n_3 - n_2)\alpha + n_3\}$ and based on the value of P_1, P_2, P_3 and P_4 determined the results of multiplication two TFNs.

For more about the arithmetic operations on fuzzy numbers, anyone can follow the article Mukherjee et al. (2023).

3. Initial valued second order linear homogeneous difference equation with constant coefficient

Let us consider the second order linear in homogeneous difference equation associated with the fuzzy initial value as

$$\begin{cases}\rho_{n+2} + d_1\rho_{n+1} + d_2\rho_n = 0 \\ \rho_0 = \tilde{p} \\ \rho_1 = \tilde{q}\end{cases} \tag{6}$$

where the coefficients d_1 and d_2 are constant.

Before solving the difference equation, we express the fuzzy initial numbers as the sum of the crisp portion and the fully uncertain portion. Therefore, $\tilde{p} = p + \tilde{p}_{un}$ (crisp part + uncertain part) and $\tilde{q} = q + \tilde{q}_{un}$.

Now we write down the Equation (6) as a two distinct part of the following problems:

I). The homogeneous crisp second order difference equation with crisp initial value as

$$\begin{cases}\rho_{n+2} + d_1\rho_{n+1} + d_2\rho_n = 0 \\ \rho_0 = p \\ \rho_1 = q\end{cases} \tag{7}$$

II). The homogeneous fuzzy second order difference equation with fuzzy initial value as

$$\begin{cases}\rho_{n+2} + d_1\rho_{n+1} + d_2\rho_n = 0 \\ \tilde{\rho}_0 = \tilde{p}_{un} \\ \tilde{\rho}_1 = \tilde{q}_{un}\end{cases} \tag{8}$$

Our actual interest is to solve Equation (8) as the crisp homogeneous problem in Equation (7) can be solved easily. The fuzzy solution of Equation (8) is Alamin et al. (2025).

$$\tilde{\rho}_n^{un} = \{\rho_n | \rho_{n+2} + d_1\rho_{n+1} + d_2\rho_n = 0, \rho_0 = p_{un}\,; \rho_1 = q_{un};\ p_{un} \in \tilde{p}_{un};\ q_{un} \in \tilde{q}_{un}\} \tag{9}$$

and the membership function of the fuzzy solution is defined as

$$\mu_{\tilde{\rho}_n^{un}}(\rho_n) = min\{\mu_{\tilde{p}_{un}}(p_{un}), \mu_{\tilde{q}_{un}}(q_{un})\} \tag{10}$$

Therefore, the fuzzy solution of Equation (6) is $\tilde{\rho}_n = \rho_n + \tilde{\rho}_n^{un}$, where ρ_n and $\tilde{\rho}_n^{un}$ are the crisp solution of Equation (7) and fuzzy solution of Equation (8), respectively.

Let the corresponding auxiliary difference equation of the homogeneous problem of Equation (7) is

$$t^2 + d_1 t + d_2 = 0 \tag{11}$$

Depending on the values of the discriminant in Equation (11), the roots may be real, equal and imaginary. Thus, we proceed along considering the three different cases separately and try to construct the general structure of the solution of Equation (6) through the geometric approach.

Case I: $(d_1)^2 - 4d_2 > 0$

The roots of Equation (11) are real and distinct. Let ξ_1 and ξ_2 are the distinct roots of Equation (11), then the linearly independent solution of Equation (7) is ${\xi_1}^n, {\xi_2}^n$.

Let the linearly independent solutions of Equation (7) are $(\rho_n^1,\ \rho_n^2)$. Then, the general solution of Equation (7) is $\rho_n = k_1\rho_n^1 + k_2\rho_n^2$ or in vector representation $\rho_n = z_n k$, where $z_n = (\rho_n^1,\ \rho_n^2)$ and $k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$. Using the crisp initial condition, we have the relation for k_1 and k_2 as

$$\begin{cases}k_1\rho_0^1 + k_2\rho_0^2 = p \\ k_1\rho_1^1 + k_2\rho_1^2 = q\end{cases} \tag{12}$$

Also, the Equation (12) can be written as $Hk = g$ where, $H = \begin{pmatrix} \rho_0^1 & \rho_0^2 \\ \rho_1^1 & \rho_1^2 \end{pmatrix}$ and $g = \begin{pmatrix} p \\ q \end{pmatrix}$. Therefore, the solution for k_1 and k_2 is obtained by

$$k = H^{-1}g \tag{13}$$

Using the value of k of Equation (13), the general solution of Equation (7) is

$$\rho_n = z_n H^{-1} g \tag{14}$$

or,

$$\rho_n = \zeta_n g = \zeta_n^1 p + \zeta_n^2 q \tag{15}$$

where,

$$\zeta_n = z_n H^{-1} \tag{16}$$

Thus, the fuzzy solution of Equation (8), in a computational way we may write as

$\tilde{\rho}_n^{un} = \left\{ \zeta_n g \middle| g = \begin{pmatrix} p_{un} \\ q_{un} \end{pmatrix};\ p_{un} \in \tilde{p}_{un};\ q_{un} \in \tilde{q}_{un} \right\}$, where the fixed vector ζ_n can be calculated easily using the Equation (16).

Note 1: (Gasilov et al., 2012) If the initial values are either triangular fuzzy numbers or in the form of parametric fuzzy numbers, then the fuzzy solution $\tilde{\rho}_n^{un} = {\zeta_1}^n \tilde{p}_{un} + {\zeta_2}^n \tilde{q}_{un}$.

Now, the casoration matrix $H_0 = \begin{pmatrix} 1 & 1 \\ \xi_1 & \xi_2 \end{pmatrix}$, then ${H_0}^{-1} = \frac{1}{\xi_2 - \xi_1} \begin{pmatrix} \xi_2 & -1 \\ -\xi_1 & 1 \end{pmatrix}$.

Therefore, using Equation (16), the fixed vector

$$\zeta_n = (\zeta_n^1, \zeta_n^2) = ({\xi_1}^n, {\xi_2}^n) {H_0}^{-1}$$
$$= ({\xi_1}^n, {\xi_2}^n) \frac{1}{\xi_2 - \xi_1} \begin{pmatrix} \xi_2 & -1 \\ -\xi_1 & 1 \end{pmatrix} \left(\frac{{\xi_1}^n \xi_2 - {\xi_2}^n \xi_1}{\xi_2 - \xi_1}, \frac{{\xi_2}^n - {\xi_1}^n}{\xi_2 - \xi_1} \right)$$

Thus, the fuzzy solution of Equation (8) is

$$\tilde{\rho}_n^{un} = \frac{{\xi_1}^n \xi_2 - {\xi_2}^n \xi_1}{\xi_2 - \xi_1} \tilde{p}_{un} + \frac{{\xi_2}^n - {\xi_1}^n}{\xi_2 - \xi_1} \tilde{q}_{un} \tag{17}$$

Case II: $(d_1)^2 - 4d_2 = 0$

The roots of Equation (11) are both equal (say ξ_1) then, the linearly independent solution of Equation (7) are ${\xi_1}^n, n{\xi_1}^n$.

Let, $z_n = (\rho_n^1,\ \rho_n^2) = ({\xi_1}^n, n{\xi_1}^n)$ be the linearly independent solution. The matrix

$$H_0 = \begin{pmatrix} \rho_0^1 & \rho_0^2 \\ \rho_1^1 & \rho_1^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \xi_1 & \xi_1 \end{pmatrix} \tag{18}$$

Therefore, ${H_0}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & \frac{1}{\xi_1} \end{pmatrix}$ and the fixed vector

$$\zeta_n = (\zeta_n^1, \zeta_n^2) = ({\xi_1}^n, \mathrm{n}{\xi_1}^n) {H_0}^{-1}$$
$$= ({\xi_1}^n, n{\xi_1}^n) \begin{pmatrix} 1 & 0 \\ -1 & \dfrac{1}{\xi_1} \end{pmatrix}$$
$$= \left({\xi_1}^n - n{\xi_1}^n, n{\xi_1}^{n-1} \right)$$

Thus the fuzzy solution of Equation (8) is

$$\tilde{\rho}_n^{un} = ({\xi_1}^n - n{\xi_1}^n) \tilde{p}_{un} + n{\xi_1}^{n-1} \tilde{q}_{un} \tag{19}$$

Case III: $(d_1)^2 - 4d_2 < 0$

The roots of Equation (11) are both complex conjugate numbers that is $\xi_{1,2} = \sigma \pm i\tau$. Then, the linearly independent solution of Equation (7) are $r^n \cos n\theta$, $r^n \sin n\theta$ where $r = \sqrt{\sigma^2 + \tau^2}$ and $\theta = \tan^{-1}\left(\frac{\tau}{\sigma}\right)$.

Let, $z_n = (\rho_n^1,\ \rho_n^2) = (r^n \cos n\theta\, , r^n \sin n\theta)$ be the linearly independent solution. The matrix

$$H_0 = \begin{pmatrix} \rho_0^1 & \rho_0^2 \\ \rho_1^1 & \rho_1^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ r\cos\theta & r\sin\theta \end{pmatrix}$$

Therefore, $H_0{}^{-1} = \frac{1}{r\sin\theta}\begin{pmatrix} r\sin\theta & 0 \\ -r\cos\theta & 1 \end{pmatrix}$ and the fixed vector

$$\zeta_n = (\zeta_n^1, \zeta_n^2) = (r^n \cos n\theta\,, r^n \sin n\theta) H_0{}^{-1}$$
$$= (r^n \cos n\theta\,, r^n \sin n\theta)\frac{1}{r\sin\theta}\begin{pmatrix} r\sin\theta & 0 \\ -r\cos\theta & 1 \end{pmatrix}$$
$$= \left(r^n \cos n\theta - \frac{r^n \sin n\theta \cos\theta}{\sin\theta}, \frac{r^n \sin n\theta}{r\sin\theta}\right)$$

Thus the fuzzy solution of Equation (8) is

$$\tilde{\rho}_n^{un} = \left(r^n \cos n\theta - \frac{r^n \sin n\theta \cos\theta}{\sin\theta}\right)\tilde{p}_{un} + \frac{r^n \sin n\theta}{r\sin\theta}\tilde{q}_{un}$$
$$= (r^n \cos n\theta - r^n \sin n\theta \cot\theta)\tilde{p}_{un} + \frac{r^{n-1}\sin n\theta}{\sin\theta}\tilde{q}_{un} \quad (20)$$

Note 2: The principal amplitude $\theta \in (-\pi, \pi) - \{0\}$. If the values of θ are either π or 0 then the roots of Equation (11) lie on the real axis only and hence we proceed through case II.

4. Illustrative example and results

This section describes the examples of fuzzy difference equations and their solutions in a fuzzy environment by geometric approch. Further analysis of the results in the geometric interface by drawing the graphical structure of the solutions are given as follows:

Example 4.1: Consider the fuzzy difference equation

$$\begin{cases} 2\rho_{n+2} - 3\rho_{n+1} + \rho_n = 0 \\ \rho_0 = (9, 10, 10.5) \\ \rho_1 = (7.2, 8, 8.7) \end{cases} \quad (21)$$

We express the initial values as the sum of the crisp part and the totally fuzzy portion as $\tilde{p} = 10 + (-1, 0, 0.5)$ and $\tilde{q} = 8 + (-0.8, 0, 0.7)$. Now, we solve the two problems separately:

The crisp problem

$$\begin{cases} 2\rho_{n+2} - 3\rho_{n+1} + \rho_n = 0 \\ \rho_0 = 10 \\ \rho_1 = 8 \end{cases} \quad (22)$$

and the fuzzy problem

$$\begin{cases} 2\rho_{n+2} - 3\rho_{n+1} + \rho_n = 0 \\ \rho_0 = (-1, 0, 0.5) \\ \rho_1 = (-0.8, 0, 0.7) \end{cases} \quad (23)$$

The auxiliary roots of the corresponding difference equation are 1 and ½ which are real and distinct. Thus, we follow case I and using Equation (19).

The crisp solution $\rho_n = 6 + 2^{-n+2}$ and the fuzzy solution is $\tilde{\rho}_n^{un} = (2^{-n+1} - 1)(-1, 0, 0.5)$ $+(2 - 2^{-n+1})(-0.8, 0, 0.7)$.

Therefore, the fuzzy solution of Equation (21) is

$$\tilde{\rho}_n = 6 + 2^{-n+2} + (2^{-n+1} - 1)(-1, 0, 0.5) + (2 - 2^{-n+1})(-0.8, 0, 0.7) \quad (24)$$

The α −cut of Equation (24) is

$$(\tilde{\rho}_n)_\alpha = 6 + 2^{-n+2} + (2^{-n+1} - 1)(1-\alpha)[-1, 0.5] + (2 - 2^{-n+1})(1-\alpha)[-0.8, 0.5] \quad (25)$$

Figure 2 shows the geometric structure of Equation (25), the solution of the fuzzy difference Equation (21).

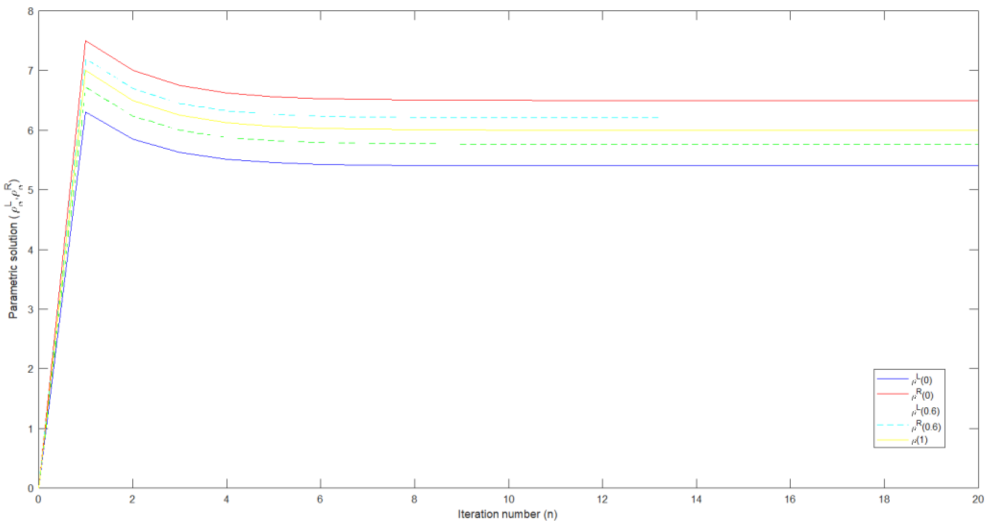

Figure 2. Graphical representation of iteration number (n) vs parametric solution $\left(\rho_n^L(\alpha), \rho_n^R(\alpha)\right)$ for $\alpha =$ 0,0.6,1, respectively

Example 4.2: Consider the fuzzy difference equation

$$\begin{cases} 4\rho_{n+2} - 20\rho_{n+1} + 25\rho_n = 0 \\ \rho_0 = (0.1, 0.3, 0.5) \\ \rho_1 = (0.6,1, 1.3) \end{cases} \tag{26}$$

We express the initial values as the sum of the crisp part and the totally fuzzy portion as $\tilde{p} = 0.3 + (-0.2, 0, 0.2)$ and $\tilde{q} = 1 + (-0.4, 0, 0.3)$. Now, we solve the two problems separately:

The crisp problem

$$\begin{cases} 4\rho_{n+2} - 20\rho_{n+1} + 25\rho_n = 0 \\ \rho_0 = 0.3 \\ \rho_1 = 1 \end{cases} \tag{27}$$

and the fuzzy problem

$$\begin{cases} 4\rho_{n+2} - 20\rho_{n+1} + 25\rho_n = 0 \\ \rho_0 = (-0.2, 0, 0.2) \\ \rho_1 = (-0.4, 0, 0.3) \end{cases} \tag{28}$$

The auxiliary roots of the corresponding difference Equation (27) are repeated by $5/2$. Thus, we follow case II and using Equation (17).

The crisp solution $\rho_n = \left(\left(\frac{5}{2}\right)^n - n\left(\frac{5}{2}\right)^n\right)0.3 + n\left(\frac{5}{2}\right)^{n-1}$ and the fuzzy solution is $\tilde{\rho}_n^{un} =$ $\left(\left(\frac{5}{2}\right)^n - n\left(\frac{5}{2}\right)^n\right)(-0.2, 0, 0.2) + n\left(\frac{5}{2}\right)^{n-1}(-0.4, 0, 0.3)$.

Therefore, the fuzzy solution of Equation (21) is

$$\tilde{\rho}_n = \left(\left(\frac{5}{2}\right)^n - n\left(\frac{5}{2}\right)^n\right)0.3 + n\left(\frac{5}{2}\right)^{n-1} + \left(\left(\frac{5}{2}\right)^n - n\left(\frac{5}{2}\right)^n\right)(-0.2, 0, 0.2) + n\left(\frac{5}{2}\right)^{n-1}(-0.4, 0, 0.3) \tag{29}$$

The α −cut of Equation (29) is

$$(\tilde{\rho}_n)_\alpha = \left(\left(\frac{5}{2}\right)^n - n\left(\frac{5}{2}\right)^n\right)0.3 + n\left(\frac{5}{2}\right)^{n-1} + \left(\left(\frac{5}{2}\right)^n - n\left(\frac{5}{2}\right)^n\right)(1-\alpha)[-0.2, 0.2] + n\left(\frac{5}{2}\right)^{n-1}(1-\alpha)[-0.4, 0.3] \tag{30}$$

Figure 3 depicts the geometric structure of Equation (30), the solution of the fuzzy difference Equation (26).

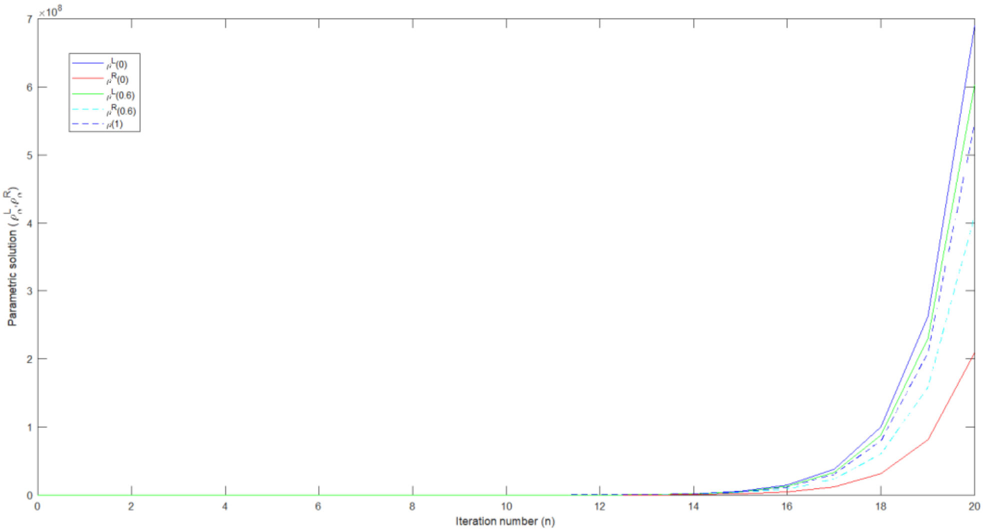

Figure 3: Graphical representation of iteration number (n) vs parametric solution $\left(\rho_n^L(\alpha), \rho_n^R(\alpha)\right)$ for $\alpha =$ 0,0.6,1, respectively

Example 4.3: Consider the fuzzy difference equation

$$\begin{cases} \rho_{n+2} - 2\rho_{n+1} + 4\rho_n = 0 \\ \rho_0 = (0.4, 0.6, 0.8) \\ \rho_1 = (1.2, 1.5, 1.7) \end{cases} \tag{31}$$

We express the initial values as the sum of the crisp part and the totally fuzzy portion as $\tilde{p} = 0.6 + (-0.2, 0, 0.2)$ and $\tilde{q} = 1.5 + (-0.3, 0, 0.2)$. Now, we solve the two problems separately:

$$\begin{cases} \rho_{n+2} - 2\rho_{n+1} + 4\rho_n = 0 \\ \rho_0 = 0.6 \\ \rho_1 = 1.5 \end{cases} \tag{32}$$

and

$$\begin{cases} \rho_{n+2} - 2\rho_{n+1} + 4\rho_n = 0 \\ \rho_0 = (-0.2, 0, 0.2) \\ \rho_1 = (-0.3, 0, 0.2) \end{cases} \tag{33}$$

The roots of Equation (32) are both complex conjugate numbers that are $\xi_{1,2} = 1 \pm i\sqrt{3}$. Then, the linearly independent solution of Equation (32) is $2^n \cos n\frac{\pi}{3}$ and $2^n \sin n\frac{\pi}{3}$.

Therefore, the crisp solution of Equation (32) is $\rho_n = 2^n \left[0.6 \cos n\frac{\pi}{3} + 0.3\sqrt{3} \sin n\frac{\pi}{3}\right]$ and the fuzzy solution of Equation (33) is

$$\tilde{\rho}_n^{un} = \left(2^n \cos\frac{n\pi}{3} - 2^n \sin\frac{n\pi}{3} \cot\frac{\pi}{3}\right)(-0.2, 0, 0.2) + \frac{2^{n-1} \sin\frac{n\pi}{3}}{\sin\frac{\pi}{3}}(-0.3, 0, 0.2)$$

$$= \left(2^n \cos\frac{n\pi}{3} - \frac{2^n}{\sqrt{3}} \sin\frac{n\pi}{3}\right)(-0.2, 0, 0.2) + \frac{2^n}{\sqrt{3}} \sin\frac{n\pi}{3}(-0.3, 0, 0.2) \tag{34}$$

Thus, the fuzzy solution Equation (31) is

$$\tilde{\rho}_n = 2^n \left[0.6 \cos n\frac{\pi}{3} + 0.3\sqrt{3} \sin n\frac{\pi}{3}\right] + \left(2^n \cos\frac{n\pi}{3} - \frac{2^n}{\sqrt{3}} \sin\frac{n\pi}{3}\right)(-0.2, 0, 0.2) + \frac{2^n}{\sqrt{3}} \sin\frac{n\pi}{3}(-0.3, 0, 0.2) \tag{35}$$

The α −cut of equation (35) is

$$(\tilde{\rho}_n)_\alpha = 2^n \left[0.6 \cos n\frac{\pi}{3} + 0.3\sqrt{3} \sin n\frac{\pi}{3}\right] + \left(2^n \cos\frac{n\pi}{3} - \frac{2^n}{\sqrt{3}} \sin\frac{n\pi}{3}\right)(1-\alpha)[-0.2, 0.2] + \frac{2^n}{\sqrt{3}} \sin\frac{n\pi}{3}(1-\alpha)[-0.3, 0.2] \tag{36}$$

Figure 4 shows the geometric structure of Equation (36), the solution of the fuzzy difference Equation (31).

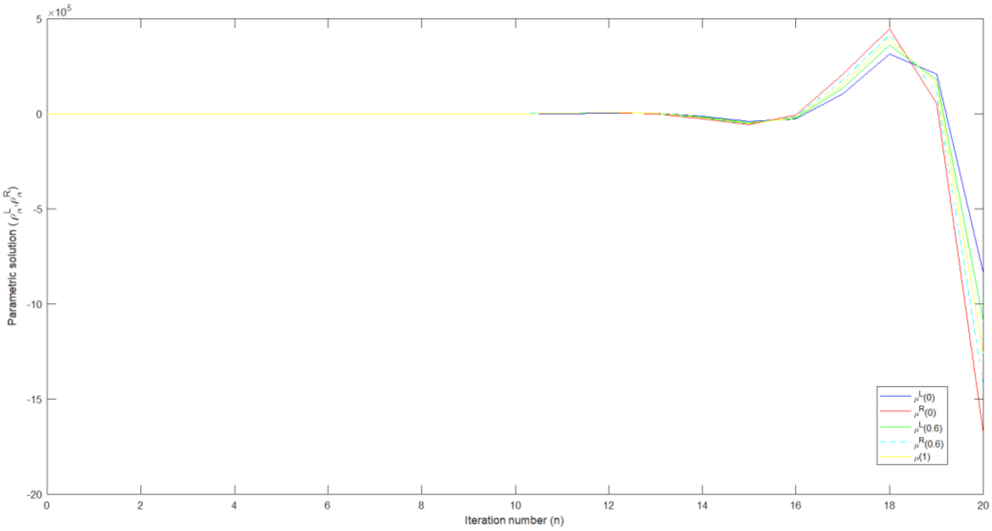

Figure 4. Graphical representation of iteration number (n) vs parametric solution $\left(\rho_n^L(\alpha), \rho_n^R(\alpha)\right)$ for $\alpha =$ 0,0.6,1, respectively

Remarks and graphical discussion: On the basis of the observation of the graphical scenarios of the above three examples, we may state the following remarkable points:

I). For $\alpha = 1$, the solution represents the crisp solutions of the corresponding problems and it is stable. However, the other two fuzzy solution branches indicate that they are stable in the fuzzy sense and the fuzzy deviation is almost constant after the second generation (see Figure 2).

II). In Figure 3, we see that the crisp solution, as well as the fuzzy solutions, are unstable. After the 14 iteration number, the solution curve increases rapidly and the fuzzy deviations with respect to the crisp solution are clearly observed.

III). Both the crisp and fuzzy solutions are unstable. The fuzziness throughout the iteration level is not uniform (see Figure 4).

5. Conclusion

We have studied the solution procedure of linear second order fuzzy initial valued homogeneous difference equation by a geometric approach based on the linear transformation method. Through this article a general outline of solution structure given theoretically. The numerical examples and their corresponding graphical explanations have been given considering the initial conditions as triangular fuzzy numbers. The solution of the discussed numerical examples are drawn directly using the theoretical knowledge of the mentioned theory. Through these numerical examples, we have verified the three different scenarios depending on the roots of the auxiliary equations.

In future, the proposed method can be extended to the initial value problem as well as the boundary value problem of the second order in homogeneous linear difference equation with constant coefficient where the initial and boundary values are fuzzy numbers with forcing factor is a fuzzy function.

References

Abid, M., & Saqlain, M. (2024). Optimizing Diabetes Data Insights Through Kmapper-Based Topological Networks: A Decision Analytics Approach for Predictive and Prescriptive Modeling. Management Science Advances, 1(1), 1-19.

Alamin, A., Mondal, S. P., Alam, S., Ahmadian, A., Salahshour, S., & Salimi, M. (2020). Solution and interpretation of neutrosophic homogeneous difference equation. Symmetry, 12(7), 1091.

Alamin, A., Rahaman, M., & Mondal, S. P. (2025). Geometric Approach for Solving First Order Non-Homogenous Fuzzy Difference Equation. Spectrum of Operational Research, 2(1), 61-71.

Ayub, S., Shabir, M., Riaz, M., Waqus, M., Bozanic, D., & Marinkovic, D. (2022). Linear Diophantine Fuzzy Rough Sets: A New Rough Set Approach with Decision Making. Symmetry, 14(3), 525.

Božanić, D., Pamučar, D., Badi, I., & Tešić, D. A decision support tool for oil spill response strategy selection: application of LBWA and Z MABAC methods. OPSEARCH 2023, 60(1), 24-58.

Chakraborty, J., Mukherjee, S., & Sahoo, L. (2024). An Alternative Approach for Enhanced Decision-Making using Fermatean Fuzzy Sets. Spectrum of Engineering and Management Sciences, 2(1), 135-150.

Deeba, E. Y., & De Korvin, A. (1999). Analysis by fuzzy difference equations of a model of CO2 level in the blood. Applied mathematics letters, 12(3), 33-40.

Gasilov, N. A., Amrahov, Ş. E., Fatullayev, A. G., & Hashimoglu, I. F. (2015). Solution method for a boundary value problem with fuzzy forcing function. Information Sciences, 317, 349-368.

Gasilov, N. A., Hashimoglu, I. F., Amrahov, S. E., & Fatullayev, A. G. (2012). A new approach to non-homogeneous fuzzy initial value problem. Computer Modeling in Engineering & Sciences, 85(4), 367-378.

Gasilov, N., Amrahov, Ş. E., & Fatullayev, A. G. (2014). Solution of linear differential equations with fuzzy boundary values. Fuzzy Sets and Systems, 257, 169-183.

Gasilov, N., Amrahov, S. G., & Fatullayev, A. G. (2011). A geometric approach to solve fuzzy linear systems of differential equations. Applied Mathematics & Information Sciences, 5(3), 484-499.

Gazi, K. H., Biswas, A., Singh, P., Rahaman, M., Maity, S., Mahata, A., & Mondal, S. P. (2024). A Comprehensive Literature Review of Fuzzy Differential Equations with Applications. Journal of Fuzzy Extension and Applications. https://doi.org/10.22105/jfea.2024.449970.1426

Hussain, A., & Ullah, K. (2024). An Intelligent Decision Support System for Spherical Fuzzy Sugeno-Weber Aggregation Operators and Real-Life Applications. Spectrum of Mechanical Engineering and Operational Research, 1(1), 177-188.

Kamran, M., Ashraf, S., Kalim Khan, S., Hussain Khan, A., Zardi, H., & Mehmood, S. (2024). Integrated Decision-Making Framework for Hospital Development: A Single-Valued Neutrosophic Probabilistic Hesitant Fuzzy Approach With Innovative Aggregation Operators. Yugoslav Journal of Operations Research, 34(3), 515-550.

Khastan, A. (2018). Fuzzy logistic difference equation. Iranian Journal of Fuzzy Systems, 15(7), 55-66.

Kizielewicz, B., & Sałabun, W. (2024). SITW Method: A New Approach to Re-identifying Multi-criteria Weights in Complex Decision Analysis. Spectrum of Mechanical Engineering and Operational Research, 1(1), 215-226.

Lakshmikantham, V., & Vatsala, A. S. (2002). Basic theory of fuzzy difference equations. Journal of Difference Equations and Applications, 8(11), 957-968.

Mahmoodirad, A., & Niroomand, S. (2023). A heuristic approach for fuzzy fixed charge transportation problem. Journal of Decision Analytics and Intelligent Computing, 3(1), 139–147.

Mukherjee, A. K., Gazi, K. H., Salahshour, S., Ghosh, A., & Mondal, S. P. (2023). A brief analysis and interpretation on arithmetic operations of fuzzy numbers. Results in Control and Optimization, 13, 100312.

Özdağoğlu, A., Keleş, M. K., & Şenefe, M. (2024). Evaluation of banks in terms of customer preferences with fuzzy SWARA and fuzzy MOORA integrated approach. Journal of Decision Analytics and Intelligent Computing, 4(1), 216–232.

Pamučar D., Božanić D., Đorović B., & Milić A. (2011a). Modelling of the fuzzy logical system for offering support in making decisions within the engineering units of the Serbian army. International Journal of the Physical Sciences, 6(3), 592-609.

Pamučar D., Božanić D., & Đorović, B. (2011b). Fuzzy logic in decision making process in the Armed Forces of Serbia. Saarbrücken: LAP Lambert Academic Publishing & Co. KG.

Pamucar, D., & Ćirović, G. (2018). Vehicle route selection with an adaptive neuro fuzzy inference system in uncertainty conditions. Decision Making: Applications in Management and Engineering, 1(1), 13-37.

Pamucar, D., Deveci, M., Schitea, D., Erişkin, L., Iordache, M., & Iordache, I. (2020). Developing a novel fuzzy neutrosophic numbers based decision making analysis for prioritizing the energy storage technologies. International journal of hydrogen energy, 45(43), 23027-23047.

Pamucar, D., Žižović, M., & Đuričić, D. (2022). Modification of the CRITIC method using fuzzy rough numbers. Decision Making: Applications in Management and Engineering, 5(2), 362-371.

Riaz, M., Hashmi, M. R., Pamucar, D., & Chu, Y. M. (2021). Spherical linear Diophantine fuzzy sets with modeling uncertainties in MCDM. Computer Modeling in Engineering & Sciences, 126(3), 1125-1164.

Sarfraz, M. (2024). Interval-value Pythagorean Fuzzy Prioritized Aggregation Operators for Selecting an Eco-Friendly Transportation Mode Selection. Spectrum of Engineering and Management Sciences, 2(1), 172-201.

Singh, P., Gazi, K. H., Rahaman, M., Basuri, T., & Mondal, S. P. (2024a). Solution strategy and associated result for fuzzy Mellin transformation. Franklin Open, 7, 100112.

Singh, P., Gazi, K. H., Rahaman, M., Salahshour, S., & Mondal, S. P. (2024b). A fuzzy fractional power series approximation and Taylor expansion for solving fuzzy fractional differential equation. Decision Analytics Journal, 10, 100402.

Tešić, D. Z., Božanić, D. I., & Puška, A. (2024). Application of multi-criteria decision making for the selection of a location for crossing a water obstacle by fording in a defense operation. Vojnotehnički glasnik/Military Technical Courier, 72(3), 1120-1146.

Wang, Y., Hussain, A., Yin, S., Ullah, K., & Božanić, D. (2024), Decision-making for solar panel selection using Sugeno-Weber triangular norm-based on q-rung orthopair fuzzy information. Frontiers Energy Research, 11, 1293623.

Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338-353.

Zadeh, L.A. (1972). A Rationale for Fuzzy Control. Journal of Dynamic Systems, Measurement and Control, 94(1), 3–4.

Zadeh, L.A. (1973). Outline of a new approach to the analysis of complex systems and decision processes. IEEE Transactions on Systems, Man, and Cybernetics, 3(1), 28–44.