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Fuzzy score and fuzzy cost SuperHyperFunction

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Abstract

A HyperFunction associates each input with a set of admissible outputs, extending conventional functions by allowing multi-valued rather than single-valued mappings. A SuperHyperFunction further generalizes this concept by employing iterated powersets for both its domain and codomain, enabling the representation of hierarchical, multi-level output structures and HyperStructural multi-valued behavior within complex systems. Although HyperFunctions and SuperHyperFunctions offer expressive tools for modeling hierarchical functional relationships, their study in the existing literature remains relatively limited. This paper extends the fuzzy score function and fuzzy cost function within the frameworks of HyperFunctions and SuperHyperFunctions and provides a concise theoretical analysis of their essential properties.

Keywords: fuzzy function, fuzzy score function, fuzzy cost function.

1. Introduction

A classical function selects exactly one value in the codomain for each element of its domain, thereby encoding a deterministic input–output rule (Bylinski, 1990). A hyperfunction generalizes this notion by assigning to each input a set of possible outputs, i.e., a multi-valued image (Fujita et al., 2025). An *n*-SuperHyperFunction goes further: it maps subsets drawn from iterated powersets to higher-level powersets, supporting hierarchical, multi-level outputs and HyperStructural behavior across domains (Smarandache, 2022). These ideas have been examined in recent studies (Jdid et al., 2025; Zheng and Wang, 2025; Chen et al., 2025; Huang et al., 2025; Al-Odhari, 2025).

Various mathematical frameworks have been developed to handle uncertainty, including the Fuzzy Set (Zadeh, 1965), Intuitionistic Fuzzy Set (Atanassov, 2012), Single-valued Neutrosophic Set (Wang et al., 2010; Xu et al., 2025; Wang and Yang, 2025), Multipolar Neutrosophic set (Naveed and Ali, 2024; Du, 2025; Wu, 2025), Complex neutrosophic set (Ali and Smarandache, 2017; Rajalakshmi et al., 2025; Saini et al., 2025), and Plithogenic Set (Smarandache, 2018; Fujita, 2025d; Amable et al., 2025; Junco, 2025). A fuzzy function maps inputs either to membership grades in [0,1] or to fuzzy sets, preserving uncertainty and enabling approximate reasoning (Demirci, 1999; Perfilieva, 2004). Well-known variants include the Intuitionistic Fuzzy Functions (Tak, 2020), Neutrosophic Functions (Hatip, 2020; Vadivel and Sundar, 2021; Basker and Said, 2023), and Plithogenic Functions (Alhasan and Abdulfatah, 2023). A fuzzy score function aggregates memberships, weights, and criteria into a single scalar to rank alternatives under vagueness and partial truth. A fuzzy cost function represents uncertain costs by fuzzy numbers, supporting decisions that minimize expected expense under imprecise, context-dependent information.

Although HyperFunctions and SuperHyperFunction provide expressive tools for modeling hierarchical functional behavior, the literature remains comparatively sparse. In this paper, we extend the fuzzy score function and fuzzy cost function within the HyperFunction and SuperHyperFunction frameworks and offer a brief analysis of their basic properties.

2. Preliminaries

This section presents the key concepts and definitions required for the discussions in this paper. Unless otherwise stated, all sets and structures considered here are assumed to be finite and simple (undirected, no loops). Note that, unless otherwise specified, n denotes a natural number.

2.1 HyperFunction and n-SuperHyperFunction

Within the study of HyperStructures (Brown et al., 1992; Davvaz and Vougiouklis, 2018; Davvaz, 2020; Agusfrianto et al., 2024) and n-SuperHyperStructures (Smarandache, 2024; Fujita, 2025b) for functions, the notions of HyperFunction and n-SuperHyperFunction were formulated by Smarandache (2022). Since then, HyperFunctions have attracted substantial attention and a variety of applications have been explored. For completeness, the essential definitions and related theorems are summarized below.

Definition 2.1 (Base Set). A (finite) base set X is the foundational set from which complex structures such as powersets and HyperStructures are derived. It is formally defined as:

$$X = \{x \mid x \text{ is an element within a specified domain}\} \quad (1)$$

Definition 2.2 (Powerset) (Fujita, 2025a). The (finite) powerset of a set X, denoted P(X), is the collection of all possible subsets of X, including both the empty set and X itself. Formally, it is expressed as:

$$P(X) = \{A \mid A \subseteq X\}. \quad (2)$$

Definition 2.3 (*n*-th Powerset) (Smarandache, 2017; Smarandache, 2022) The (finite) n-th powerset of a set X, denoted $P_n(X)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(X) = P(X);\ P_{n+1}(X) = P(P_n(X)),\ \text{for } n \geq 1 \quad (3)$$

Similarly, the(finite) n-th non-empty powerset, denoted $P^*_n(X)$, is defined recursively as:

$$P^*_1(X) = P(X) \setminus \{\emptyset\};\ P^*_{n+1}) (X) = P(P^*_n(X)) \setminus \{\emptyset\} \quad (4)$$

Example 2.4: Let $X = \{Alice, Bob, Carol\}$ be the set of three employees in an IT company.

1. $P_1(X) = P(X)$ is the set of all possible teams that can be formed from these employees: $P_1(X)$ = {Ø, {Alice}, {Bob}, {Carol}, {Alice, Bob}, {Alice, Carol}, {Bob, Carol}, {Alice, Bob, Carol}}.
2. P2(X) = P(P1(X)) is the set of all possible "collections of teams". For example, {{Alice}, {Bob, Carol}} ∈ P2(X) means: "one task is done by Alice alone, and another task is done by the team {Bob, Carol}." Thus P2(X) can model one project made of several subteams.

In practice, P1(X) corresponds to "all possible teams from the company," while P2(X) corresponds to "all possible ways to group those teams into a multi-team project." Higher powersets P3(X), P4(X), … can then represent portfolios of projects, or layers of management that group projects into larger programs.

Definition 2.5 (HyperOperation) (Spartalis, 1996; Vougiouklis, 2005; Leoreanu-Fotea et al., 2015)**:** A (finite) HyperOperation is a generalization of a binary operation where the result of combining two elements is a set, not a single element. Formally, for a set X, a HyperOperation ∘ is defined as:

$$\circ: X \times X \to P(X) \quad (5)$$

Definition 2.6 (HyperFunction) (Fujita et al., 2025; Fujita, 2025c)**.** A (finite) HyperFunction is a function where the domain remains a classical set X, but the codomain is extended to the powerset of X, denoted $P(X)$. Formally, a HyperFunction f is defined as:

$f: X \to P(X)$. For any $x \in X$, $f(x) \subseteq X$ is a subset of X (6)

Example 2.7 : Let $X = \{email, chat, phone\}$ be the possible contact channels for a customer. Define

f: X → P(X) by

f(email) = {email, chat}, f(chat) = {chat}, f(phone) = {phone, email}.

This models a helpdesk rule: if a customer writes by email, the system may respond by email and additionally open a chat; if the customer calls by phone, the system may register both a phone response and a confirmation email. One input (a single channel) is mapped to several possible output channels (a set), so *f* is a HyperFunction.

Definition 2.8 (SuperHyperOperations) (Fujita, 2025b). Let *X* be a (finite) non-empty set, and let $P_k(X)$ be the (finite) *k*-th powerset of *X*. Define

$$P^0(H) = H, \qquad P^{k+1}(H) = P(P^k(H)) \text{ for } k \geq 0 \quad (7)$$

A (finite) SuperHyperOperation of order *(m, n)* is an *m*-ary operation

$$\circ^{(m, n)}: H^m \to P^*_n(X) \quad (8)$$

If the codomain excludes the empty set, it is classical-type; if it includes it, it is Neutrosophic-type.

Definition 2.9 (*n*-SuperHyperFunction) (Smarandache, 2022). An (finite) *n*-SuperHyperFunction generalizes the concept of a HyperFunction by using the *n*-th powerset $P_n(S)$ as the codomain. Formally, for $n \geq 2$, an *n*-SuperHyperFunction f is defined as:

$$f: P_r(S) \to P_n(S) \quad (9)$$

where $0 \leq r \leq n$, and $P_n(S)$ is the *n*-th powerset of *S*. This definition allows *f* to map subsets of *S* (from $P_r(S)$) to elements in the *n*-th powerset $P_n(S)$.

Example 2.10.: Let $S = \{HR, Accounting, IT\}$ be the departments of a company. A request may involve several departments at once (for example *{HR, IT}*), and each such joint request must be expanded into several "bundles" of subworkflows (for example a bundle for approval, a bundle for security checks, and a bundle for payroll update). Define

g: P(S) → $P_2(S)$

that sends a subset like {HR, IT} to a collection of subsets of P(S), for example

g({HR, IT}) = {{{HR}, {IT}}, {{HR, Accounting}}}.

This means: the joint request {HR, IT} will be realized as (i) one workflow where HR and IT act separately, and (ii) an alternative workflow where HR must cooperate with accounting. Since the codomain is the second powerset $P_2(S)$, g is an n-SuperHyperFunction with n = 2.

2.2. Fuzzy score function

A score function maps an object to a single scalar, enabling total ordering and comparison across alternatives under defined criteria (Nancy and Garg, 2016). A fuzzy score function assigns each fuzzy quantity a scalar, aggregating membership information to rank uncertain alternatives consistently and comparably (Kumar et al., 2025).

Definition 2.11 (Lee, 2024): A fuzzy score function is a mapping $S: F \to R$ that assigns each fuzzy quantity (e.g., a fuzzy number or an element of an extended fuzzy set model) a single scalar so that alternatives can be totally or quasi-totally ordered under uncertainty. Good score functions are designed to increase with "degree of membership" and, in models that include non-membership, to decrease with "degree of non-membership," often normalizing to *[0,1]* for comparability.

Example 2.12: Suppose two suppliers A and B are evaluated by triangular fuzzy numbers (TFNs):

$Q_A = TFN(0.6, 0.8, 0.9)$

$Q_B = TFN(0.5, 0.7, 0.95)$

Define the fuzzy score function S for a triangular fuzzy number TFN(a, b, c) by

$S(TFN(a, b, c)) = (a + b + c) / 3.$

Then

$S(Q_A) = (0.6 + 0.8 + 0.9) / 3 = 2.3 / 3 \approx 0.7667,$

$S(Q_B) = (0.5 + 0.7 + 0.95) / 3 = 2.15 / 3 \approx 0.7167.$

Since 0.7667 > 0.7167, supplier A ranks above supplier B under uncertainty.

2.3. Fuzzy cost function

A cost function assigns each decision or model a numeric penalty, measuring error or expense to guide optimization and comparison. A fuzzy cost function assigns fuzzy costs to alternatives, aggregating uncertain criteria and enabling ranking through defuzzification or fuzzy ordering (Garcia et al., 2009).

Definition 2.13 (Vasant et al., 2006; Jain and Sanga, 2020): A fuzzy cost function maps each decision, path, or design xxx to a fuzzy valuation $C(x) \in R^{\sim}$ (or to a fuzzy rule–based score), aggregating uncertain criteria such as distance, risk, or energy through fuzzy membership functions and inference. Decisions are then compared by defuzzifying or ranking these fuzzy costs. In practice, such functions have been used to evaluate and choose robot paths by fusing multiple uncertain factors into a single fuzzy “cost” for optimization.

Example 2.14: A mobile robot must choose between two paths P_1 and P_2. Each path has an uncertain (normalized) distance and risk modeled by triangular fuzzy numbers. Distances (in km) are normalized by the scale *12* km.

Path P_1: distance $D_1 = TFN(8, 10, 12) / 12 = TFN(2/3, 5/6, 1) \approx TFN(0.6667, 0.8333, 1.0000)$, risk $R_1 = TFN(0.2, 0.3, 0.5)$.

Path P_2:distance $D_2 = TFN(6, 9, 11) / 12 = TFN(0.5, 0.75, 11/12) \approx TFN(0.5000, 0.7500, 0.9167)$, risk $R_2 = TFN(0.3, 0.4, 0.6)$.

Define a fuzzy cost function with weights $w_d = 0.7$ *(distance)* and $w_r = 0.3$ *(risk)* using TFN arithmetic:

Scalar multiply: $\alpha \cdot TFN(a, b, c) = TFN(\alpha a, \alpha b, \alpha c)$.

Componentwise add: $TFN(a_1, b_1, c_1) \oplus TFN(a_2, b_2, c_2) = TFN(a_1 + a_2, b_1 + b_2, c_1 + c_2)$.

Set $C(P) = w_d \cdot D \oplus w_r \cdot R$.

Compute: $C(P_1) = 0.7 \cdot TFN(0.6667, 0.8333, 1.0000) \oplus 0.3 \cdot TFN(0.2, 0.3, 0.5)$

$= TFN(0.4667, 0.5833, 0.7000) \oplus TFN(0.0600, 0.0900, 0.1500)$

$= TFN(0.5267, 0.6733, 0.8500)$.

$C(P_2) = 0.7 \cdot TFN(0.5000, 0.7500, 0.9167) \oplus 0.3 \cdot TFN(0.3, 0.4, 0.6)$

$= TFN(0.3500, 0.5250, 0.6417) \oplus TFN(0.0900, 0.1200, 0.1800)$

$= TFN(0.4400, 0.6450, 0.8217)$.

Defuzzify by the centroid: $cen(TFN(a, b, c)) = (a + b + c) / 3$.

$cen(C(P_1)) = (0.5267 + 0.6733 + 0.8500) / 3 \approx 0.6833,$

$cen(C(P_2)) = (0.4400 + 0.6450 + 0.8217) / 3 \approx 0.6356.$

Because $0.6356 < 0.6833$, the fuzzy cost function selects P_2 as the better (lower-cost) path under uncertainty.

3. Main Results

This section presents the main contributions of the paper.

3.1. Fuzzy Score HyperFunction

A Fuzzy Score HyperFunction assigns each fuzzy object the set of all admissible scalar scores, capturing modeling or parameter uncertainty.

Definition 3.1.: Fix a nonempty family S of admissible fuzzy score maps on F, i.e.,

$$S \subseteq \{s: F \to R\} \text{ with } S \neq \emptyset \quad (10)$$

where each $s \in S$ satisfies the modeling requirements for a score (for example, it is nondecreasing in "membership degree," nonincreasing in "non-membership," and optionally normalized to *[0,1]*).

Define the Fuzzy Score HyperFunction associated with S by

$$H_S: F \to P^*(R) \tag{11}$$

given by

$$H_S(x) := \{s(x): s \in S\} \text{ for every } x \in F \tag{12}$$

Intuition: $H_S(x)$ collects all admissible scalar scores of x produced by the family S (e.g., different defuzzification rules, parameter settings, or weighting schemes). Thus, H_S is multi-valued in the codomain and captures score uncertainty or modeling ambiguity.

Example 3.2.: Let F be the set of triangular fuzzy numbers $TFN(a, b, c)$ with $a \le b \le c$ in $[0,1]$. For $\lambda \in [0,1]$, define an admissible score: $s_\lambda(TFN(a, b, c)) := (a + \lambda b + c) / (2 + \lambda)$. Interpretation: λ interpolates between the simple average of the endpoints $(\lambda = 0)$ and the usual centroid $(\lambda = 1)$. Set $S := \{ s_\lambda : \lambda \in [0,1] \}$ and define H_S accordingly.

Compare two suppliers with quality TFNs:

$Q_A = TFN(0.6, 0.8, 0.9)$

$Q_B = TFN(0.5, 0.7, 0.95)$

For Q_A:

$s_0(Q_A) = (0.6 + 0.9) / 2 = 0.75$

$s_1(Q_A) = (0.6 + 0.8 + 0.9) / 3 = 2.3 / 3 \approx 0.7667$

Hence $H_S(Q_A) \supseteq [0.75, 0.7667]$ (indeed, the full image over $\lambda \in [0,1]$ is a compact interval between these endpoints).

For Q_B:

$s_0(Q\_B) = (0.5 + 0.95) / 2 = 0.725$

$s_1(Q\_B) = (0.5 + 0.7 + 0.95) / 3 = 2.15 / 3 \approx 0.7167$

Hence H_S(Q_B) ⊇ [0.7167, 0.725] (again an interval).

Dominance: note that min $H_S(Q_A) = 0.75 > 0.725 = \max H_S(Q_B)$. Therefore, Q_A robustly outranks Q_B for all $\lambda \in [0,1]$. This illustrates how the HyperFunction returns a set of plausible scores, while still enabling clear decisions when the sets are separated.

Theorem 3.3.: For any nonempty family $S \subseteq \{s: F \to R\}$, the map $H_S: F \to P^*(R)$ defined by $H_S(x) = \{s(x): s \in S\}$ is a HyperFunction.

Proof. Fix $x \in F$. Since $S \neq \emptyset$ and every $s \in S$ maps x to a real number $s(x) \in R$, the image set $H_S(x) = \{s(x): s \in S\}$ is a nonempty subset of R. Hence $H_S(x) \in P^*(R)$. Because this holds for every $x \in F$, H_S is a function from F to $P^*(R)$, i.e., a HyperFunction.

Theorem 3.4.: (i) For every (crisp) fuzzy score function $s: F \to R$, there exists a Fuzzy Score HyperFunction H_S whose values are singletons and which "embeds" s. (ii) Conversely, any single-valued selection from a Fuzzy Score HyperFunction yields a (crisp) fuzzy score function.

Proof. (i) Take $S = \{s\}$. Then for each $x \in F$, $H_S(x) = \{s(x)\}$, which is a singleton subset of R. Thus s can be recovered by the canonical "projection" *s(x) = the unique element of $H_S(x)$.* Hence every ordinary fuzzy score function is a special case of the Fuzzy Score HyperFunction.

Proof. (ii) Let S be any nonempty family and H_S the associated HyperFunction. Suppose we choose, for each $x \in F$, one element $\sigma(x) \in H_S(x)$ (i.e., a single-valued choice). Then the map $s_\sigma: F \to R$ defined by $s_\sigma(x) = \sigma(x)$ is an ordinary fuzzy score function (it assigns exactly one scalar to each x). Therefore, single-valued selections of H_S recover (crisp) fuzzy score functions.

3.2. Fuzzy Score SuperHyperFunction

A Fuzzy Score SuperHyperFunction maps fuzzy subsets to higher-level nested score families, aggregating multi-stage evaluations and preserving non-emptiness across levels.

Notation 3.5. (Singleton nesting (level raising)): For any set Z and integer $t \geq 0$ define $Nest^t(Z)$ by:

$Nest^0(Z) := Z$

$Nest^{t+1}(Z) := \{Nest^t(Z)\}$

We will use the following simple facts (by construction):

If $Z \in P_j(Y)$ and $t \geq 0$, then $Nest^t(Z) \in P_{j+t}(Y)$.

If $S \neq \emptyset$, then {something depending on $s \in S$} is nonempty.

Definition 3.6.: Let S be a nonempty family of admissible fuzzy score maps on F, i.e.,

$$S \subseteq \{s: F \to R\} \text{ with } S \neq \emptyset \quad (13)$$

Fix integers m,n with n ≥ m+1 (so at least one level of "hyper" aggregation appears).

Define the Fuzzy Score SuperHyperFunction associated with S by

$$H_S^{(m,n)}: P_m(F) \to P_n(R) \quad (14)$$

For $X \in P_m(F)$, set

$$C_S^{(m)}(X) := \{P_{m(s)}(X) : s \in S\} \in P_{m+1}(R) \quad (15)$$

and then

$$H_S^{(m,n)}(X) := Nest^{n-(m+1)}(C_S^{(m)}(X)) \in P_n(R) \quad (16)$$

Intuition:

- $P_m(s)$ applies "score s" pointwise at level m (via direct image), producing an element of $P_m(R)$.
- Collecting over all $s \in S$ yields $C_S^{(m)}(X) \in P^{m+1}(R)$.
- Additional Nest raises the level to n, encoding the super-hyper structure.

Because $S \neq \emptyset$, each $H_S^{(m,n)}(X)$ is nonempty.

Example 3.7. (Level lift with *m = 1* and *n = 2*): Let F be the set of triangular fuzzy numbers $TFN(a,b,c)$ with $0 \leq a \leq b \leq c \leq 1$. For $\lambda \in [0,1]$, define an admissible score $s_\lambda(TFN(a,b,c)) := (a + \lambda b + c) / (2 + \lambda)$. Let $S := \{s_\lambda : \lambda \in \{0, 1\}\}$ (two scoring rules: endpoints-average and centroid).

Take a finite set $X \in P_1(F)$ of candidates, e.g., $X = \{TFN(0.6,0.8,0.9), TFN(0.5,0.7,0.95)\}$. Compute $P_1(s_\lambda)(X) = \{ s_\lambda(x) : x \in X \} \subseteq R$, so $P_1(s_\lambda)(X) \in P_1(R)$. Thus $C_S^{(1)}(X) = \{P_1(s_0)(X), P_1(s_1)(X)\} \in P_2(R)$, and for $n = 2$ we have $H_S^{(1,2)}(X) = Nest^{2-(1+1)}(C_S^{(1)}(X)) = C_S^{(1)}(X) \in P_2(R)$.

So $H_S^{(1,2)}$ returns a set of (score-sets), one per scoring rule, i.e., a *2*-level super-hyper score.

Theorem 3.8.: For any nonempty $S \subseteq \{s: F \to R\}$ and integers m,n with $n \geq m+1$, the map $H_S^{(m,n)} : P_m(F) \to P_n(R)$ defined above is a *(m,n)*–SuperHyperFunction.

Proof. Fix $X \in P_m(F)$. For each $s \in S$, $P_m(s)(X) \in P_m(R)$ by the definition of the direct image lift. Therefore $C_S^{(m)}(X) := \{P_m(s)(X) : s \in S\} \subseteq P_m(R)$, hence $C_S^{(m)}(X) \in P_{m+1}(R)$. Since $n \geq m+1$, applying $Nest^{n-(m+1)}$ maps $P_{m+1}(R)$ into $P_n(R)$. Thus $H_S^{(m,n)}(X) \in P_n(R)$. As $S \neq \emptyset$, $C_S^{(m)}(X)$ is nonempty, and so is $H_S^{(m,n)}(X)$. Therefore $H_S^{(m,n)}$ is a well-typed function $P_m(F) \to P_n(R)$, i.e., a (m,n)–SuperHyperFunction.

Theorem 3.9.: The Fuzzy Score SuperHyperFunction strictly generalizes the Fuzzy Score HyperFunction. In particular, $H_S^{(0,1)} = H_S$.

Proof. Set m = 0 and n = 1. Then $P_0(F) = F$ and, for $X \in F$, $P_0(s)(X) = s(X) \in R$. Hence $C_S^{(0)}(X) = \{P_0(s)(X): s \in S\} = \{s(X): s \in S\} \subseteq R$, so $C_S^{(0)}(X) \in P_1(R) = P(R)$. Because $n-(m+1) = 1-(0+1) = 0$, we have $H_S^{(0,1)}(X) = Nest^0(C_S^{(0)}(X)) = C_S^{(0)}(X) = \{s(X) : s \in S\} = H_S(X)$. Thus the classical Fuzzy Score HyperFunction is the special case (m,n) = (0,1). Moreover, whenever S = {s} is a singleton, $H_S^{(0,1)}(x) = \{s(x)\}$ is singleton-valued, recovering the degenerate crisp score as a subcase. Therefore the SuperHyperFunction strictly generalizes the HyperFunction.

3.3. Fuzzy Cost HyperFunction

A Fuzzy Cost HyperFunction maps each item to the nonempty set of fuzzy cost values generated by alternative admissible models.

Definition 3.10.: Let Θ be a nonempty family of admissible fuzzy cost models on X, that is $\Theta \subseteq \{C: X \to FN(R)\}$ and $\Theta \neq \emptyset$. Define the Fuzzy Cost HyperFunction associated with Θ by $H_\Theta : X \to P^*(FN(R))$.

For each $x \in X$, set $H_\Theta(x) := \{C(x): C \in \Theta\}$. (Thus $H_\Theta(x)$ is the nonempty set of all fuzzy costs produced at *x* by the models in *Θ*.)

If a defuzzifier $d : FN(R) \to R$ is fixed, also define the defuzzified Fuzzy Cost HyperFunction by $H_\Theta^d : X \to P^*(R)$ $H_\Theta^d(x) := \{d(C(x)): C \in \Theta\}$.

Example 3.11.: Let $X = \{P_1, P_2\}$ be two candidate robot paths. For each path *P,* define a fuzzy distance $D_P \in FN(R)$ and a fuzzy risk $R_P \in FN(R)$. For a weight parameter $w \in [0,1]$, define a fuzzy cost model $C^{(w)}(P)= w \otimes D_P \oplus (1 - w)\otimes R_P$, where "⊗" is scalar multiplication of fuzzy numbers and "⊕" is componentwise addition (e.g., for triangular/trapezoidal fuzzy numbers).

Take $\Theta = \{C^{(0.7)}, C^{(0.5)}\}$. Then the Fuzzy Cost HyperFunction is $H_\Theta(P) = \{C^{(0.7)}(P), C^{(0.5)}(P)\} \subseteq FN(R)$, so H_Θ(P) explicitly captures the model uncertainty stemming from the weighting choice. If we fix the centroid defuzzifier *d*, the defuzzified HyperFunction becomes $H_\Theta^d(P) = \{d(C^{(0.7)}(P)), d(C^{(0.5)}(P))\} \subseteq R$, representing the set of plausible crisp costs for path P.

Theorem 3.12.: With $\Theta \neq \emptyset$ as above, $H_\Theta: X \to P^*(FN(R))$ is a hyperfunction.

Proof. For any fixed $x \in X$ and $C \in \Theta$, we have $C(x) \in FN(R)$. Therefore $H_\Theta(x) = \{C(x) : C \in \Theta\} \subseteq FN(R)$. Because $\Theta\neq\emptyset$, the set $H_\Theta(x)$ is nonempty, hence $H_\Theta(x) \in P^*(FN(R))$. Thus, H_Θ is a function from *X* into $P^*(FN(R))$, i.e., a HyperFunction.

Theorem 3.13.: The Fuzzy Cost HyperFunction strictly generalizes the ordinary fuzzy cost function. In particular, if Θ is a singleton $\{C_0\}$ with $C_0 : X \to FN(R)$, then $H_{\{C_0\}}(x) = \{C_0(x)\}$ for all $x \in X$, so $H_{\{C_0\}}$ is exactly the point-embedding of C_0 into $P^*(FN(R))$. Moreover, for any defuzzifier *d,* $H_{\{C_0\}}{}^d(x) = \{d(_{C_0(x)})\}$ embeds the crisp cost $d\circ C_0$ into $P^*(R)$.

Proof. If $\Theta = \{C_0\}$, then for each $x \in X$, $H_{\{C_0\}}(x) = \{ C(x): C \in \{C_0\} \} = \{ C_0(x) \}$. Thus $H_{\{C_0\}}$ equals the map $x \mapsto \{C_0(x)\}$, i.e., the canonical inclusion of C_0 into the HyperFunction codomain. Likewise, $H_{\{C_0\}}{}^d(x) = \{d(C(x)): C \in \{C_0\}\} = \{d(C_0(x))\}$, which is the canonical inclusion of the crisp cost $d\circ C_0$ into $P^*(R)$. Hence the Fuzzy Cost HyperFunction reduces to the ordinary fuzzy cost function (or its defuzzified version) when *Θ* is a singleton, and therefore strictly generalizes it when $|\Theta| > 1$.

3.4. Fuzzy Cost SuperHyperFunction

A Fuzzy Cost SuperHyperFunction lifts cost evaluation to higher powerset levels, aggregating model-dependent fuzzy costs into nested multi-stage structures representations.

Definition 3.14.(Fuzzy Cost HyperFunction): Let *Θ* be a nonempty family of admissible fuzzy cost models on *X,* that is $\Theta \subseteq \{C : X \to FN(R)\}$ and $\Theta \neq \emptyset$. Define the Fuzzy Cost HyperFunction associated with *Θ* by $H_\Theta: X \to P^*(FN(R))$

For each $x \in X$, set $H_\Theta(x):=\{ C(x) : C \in \Theta\}$. (Thus H_Θ(x) is the nonempty set of all fuzzy costs produced at x by the models in Θ.)

If a defuzzifier d: FN(R) → R is fixed, also define the defuzzified Fuzzy Cost HyperFunction by $H_\Theta : X \to P^*(R)$ $H_\Theta^d(x):= \{d(C(x)) : C \in \Theta\}$.

Example 3.15.: Let $X = \{P_1, P_2\}$ be two candidate robot paths. For each path *P,* define a fuzzy distance $D_P \in FN(R)$ and a fuzzy risk $R_P \in FN(R)$. For a weight parameter $w \in [0,1]$, define a fuzzy cost model $C^{(w)}(P):= w \otimes D_P \oplus (1 - w) \otimes R_P$, where "⊗" is scalar multiplication of fuzzy numbers and "⊕" is componentwise addition (e.g., for triangular/trapezoidal fuzzy numbers).

Take $\Theta = \{C^{(0.7)}, C^{(0.5)}\}$. Then the Fuzzy Cost HyperFunction is $H_\Theta(P) = \{C^{(0.7)}(P), C^{(0.5)}(P)\} \subseteq FN(R)$, so $H_\Theta(P)$ explicitly captures the model uncertainty stemming from the weighting choice. If we fix the centroid defuzzifier d, the defuzzified HyperFunction becomes $H_\Theta{}^d(P) = \{d(C^{(0.7)}(P)), d(C^{(0.5)}(P))\} \subseteq R$, representing the set of plausible crisp costs for path P.

Theorem 3.16.: With $\Theta \neq \emptyset$ as above, $H_\Theta : X \to P^*(FN(R))$ is a HyperFunction.

Proof. For any fixed $x \in X$ and $C \in \Theta$, we have $C(x) \in FN(R)$. Therefore $H_\Theta(x) = \{C(x) : C \in \Theta\} \subseteq FN(R)$. Because $\Theta \neq \emptyset$, the set $H_\Theta(x)$ is nonempty, hence $H_\Theta(x) \in P^*(FN(R))$. Thus, H_Θ is a function from X into $P^*(FN(R))$, i.e., a HyperFunction.

Theorem 3.17.: The Fuzzy Cost HyperFunction strictly generalizes the ordinary fuzzy cost function. In particular, if Θ is a singleton $\{C_0\}$ with $C_0 : X \to FN(R)$, then

$H_{\{C_0\}}(x) = \{C_0(x)\}$ for all $x \in X$, so $H_{\{C_0\}}$ is exactly the point-embedding of C_0 into $P^*(FN(R))$. Moreover, for any defuzzifier d, $H_{\{C_0\}}{}^d(x) = \{d(C_0(x))\}$ embeds the crisp cost $d \circ C_0$ into $P^*(R)$.

Proof. If $\Theta = \{C_0\}$, then for each $x \in X$, $H_{\{C_0\}}(x) = \{C(x): C \in \{C_0\}\} = \{C_0(x)\}$. Thus H_{{C₀}} equals the map $x \mapsto \{C_0(x)\}$, i.e., the canonical inclusion of C_0 into the HyperFunction codomain. Likewise, $H_{\{C_0\}}{}^d(x) = \{d(C(x)) : C \in \{C_0\}\} = \{d(C_0(x))\}$, which is the canonical inclusion of the crisp cost $d \circ C_0$ into $P^*(R)$. Hence the Fuzzy Cost HyperFunction reduces to the ordinary fuzzy cost function (or its defuzzified version) when Θ is a singleton, and therefore strictly generalizes it when $|\Theta| > 1$.

4. Conclusion

This paper has extended the fuzzy score function and fuzzy cost function within the frameworks of HyperFunctions and SuperHyperFunctions and provides a concise theoretical analysis of their essential properties.

I would like to explore possible extensions in the future using Neutrosophic Functions and Plithogenic Functions.

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